Last time

- Existence and uniqueness of solution to

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

Today: Ch. 3.2, 3.4

- Continuous dependence on initial condition and parameters.

nominal: $\dot{x}(t) = f(t, x(t), \lambda_0), \quad x(t_0) = x_0$

perturbed: $\dot{y}(t) = f(t, y(t), \lambda), \quad y(t_0) = y_0$

$$\text{if } \begin{matrix} y_0 \to x_0 \\ \text{and } \lambda \to \lambda_0 \end{matrix} \implies y(t) \to x(t) \quad (*)$$

- We prove a similar result, which can be used to prove $(*)$ (Thm. 3.5 in Khalil)

- It is a practical tool to control the error in I.C. and dynamics.

- In order to prove the thm, we need a lemma which is also useful in general.

Lemma (Bellman - Gronwall ineq.) (Appendix A)

- Assume $Z(t)$ satisfies

$$Z(t) \leq \lambda(t) + \int_{t_0}^{t} \mu(s) Z(s) ds$$

where $\mu(t) \geq 0$, $\lambda(t)$ are continuous in t.

Then,

$$Z(t) \leq \lambda(t) + \int_{t_0}^{t} e^{\int_s^t \mu(\tau) d\tau} \lambda(s) \mu(s) ds$$

Corollary: (special cases)

- if $\lambda(t) = \lambda$ const., then

$$Z(t) \leq \lambda e^{\int_{t_0}^{t} \mu(s) ds}$$

- If $\mu(t) = \mu$ const., then

$$Z(t) \leq \lambda e^{\mu(t - t_0)}$$

proof of Lemma:

- define $$\psi(t) = \int_{t_0}^{t} \mu(s) z(s)\, ds$$

- Then $$z(t) \leq \lambda(t) + \psi(t) \quad (*)$$

- our goal is to obtain a bound on $\psi(t)$

- Taking the derivative of $\psi(t)$

$$\dot{\psi}(t) = \mu(t) z(t)$$

$$\overset{(*)}{\leq} \mu(t)\psi(t) + \mu(t)\lambda(t)$$

($\mu(t) \geq 0$ is used here)

- Therefore, we have a differential inequality,

$$\dot{\psi}(t) \leq \mu(t)\psi(t) + \mu(t)\lambda(t), \quad \psi(t_0) = 0$$

- Consider the same equation, but with equality

$$\dot{x}(t) = \mu(t) x(t) + \mu(t)\lambda(t), \quad x(t_0) = 0$$

$(**)$

- We can argue that $\psi(t) \le X(t)$.

- If it is true, then $z(t) \le \lambda(t) + X(t)$

where $X(t)$ solves the linear system (**)

$$\Rightarrow X(t) = \underbrace{X(t_0)}_{0} + \int_{t_0}^{t} \underbrace{\phi(t,s)}_{\text{state transition func.}} \lambda(s)\mu(s)\, ds$$

$$\phi(t,s) = e^{\int_s^t \mu(\tau) d\tau}$$

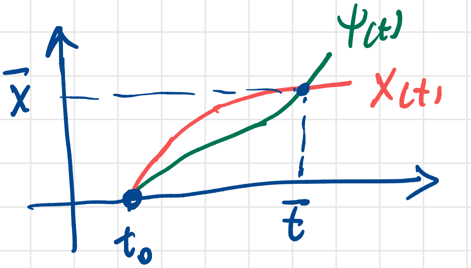
- This proves the thm. It remains to prove $\psi(t) \le X(t)$.

$$\dot{\psi} \le f(t,\psi), \quad \psi(t_0) = 0$$
$$\dot{X} = f(t,X), \quad X(t_0) = 0$$

- pictorial proof

$X(t_0) = \psi(t_0)$

$\Rightarrow \dot{X}(t_0) \ge \dot{\psi}(t_0)$



- if $\psi(t) \ge X(t)$ for some t, then $\exists\, \bar{t}$ s.t

$X(\bar{t}) = \psi(\bar{t}) \Rightarrow \dot{X}(\bar{t}) \ge \dot{\psi}(\bar{t}) \Rightarrow X(t) \ge \psi(t)$

for t close to $\bar{t}$

Contradiction $\Leftarrow$

Thm:

- Assume $f(t,x)$ is piecewise continuous in $t \in [t_0, t_1]$ and Lip. in x on the set $W \subset \mathbb{R}^n$ with Lip. constant L.

- Let $x(t), y(t)$ be solutions of

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0$$

$$\dot{y}(t) = f(t, y(t)) + g(t, y(t)), \quad y(t_0) = y_0$$

where

$$\|g(t,y)\| \leq \mu \quad \forall y \in W, \ \forall t \in [t_0, t_1]$$

- Then

$$\| y(t) - x(t) \| \leq e^{L(t-t_0)} \|y_0 - x_0\| + \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right)$$

Remark:

- It is easy to see that as $y_0 \to x_0$ and $\mu \to 0$ then $y(t) \to x(t)$

proof of the thm:

- The integral form of the solutions are

$$Y(t) = Y_0 + \int_{t_0}^{t} f(s, Y(s)) + g(s, Y(s)) \, ds$$

$$X(t) = X_0 + \int_{t_0}^{t} f(s, X(s)) \, ds$$

$$\Rightarrow \|Y(t) - X(t)\| \leq \|Y_0 - X_0\| + \int_{t_0}^{t} \| f(s, Y(s)) - f(s, X(s)) \| \, ds$$

$$+ \int_{t_0}^{t} \| g(s, Y(s)) \| \, ds$$

$$\leq \|Y_0 - X_0\| + \int_{t_0}^{t} L \| Y(s) - X(s) \| \, ds + \mu (t - t_0)$$

- Apply B-G lemma to $Z(t) = \|Y(t) - X(t)\|$

$$\|Y(t) - X(t)\| \leq \|Y_0 - X_0\| + \mu (t - t_0) + \underbrace{\int_{t_0}^{t} L \left[\|Y_0 - X_0\| + \mu (s - t_0) \right] e^{(t-s)L} \, ds}_{\text{Consider this term}}$$

- Integration by parts:

$$\int_{t_0}^{t} \left[\|Y_0 - X_0\| + \mu(s-t_0)\right]\left(-\frac{d}{ds} e^{(t-s)L}\right) ds$$

$$= -\left[\|Y_0 - X_0\| + \mu(s-t_0)\right] e^{(t-s)L} \Big|_{t_0}^{t}$$

$$+ \int_{t_0}^{t} \mu e^{(t-s)L} ds$$

$$= -\|Y_0 - X_0\| - \mu(t-t_0) + \|Y_0 - X_0\| e^{(t-t_0)L}$$

$$+ \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right)$$

$\Longrightarrow$

$$\|Y(t) - X(t)\| \leq \|Y_0 - X_0\| e^{(t-t_0)L} + \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right)$$

Example:

$$\dot{x}(t) = -x(t), \qquad x(0) = x_0$$
$$\dot{y}(t) = -y(t), \qquad y(0) = y_0$$

- From thm.

$$|x(t) - y(t)| \leq e^{Lt} |x_0 - y_0| \qquad \text{with } L = 1$$

- From explicit form of solution

$$|x(t) - y(t)| = |x_0 e^{-t} - y_0 e^{-t}|$$
$$= e^{-t} |x_0 - y_0|$$

- This is much smaller than the bound $e^{t} |x_0 - y_0|$ when t is large.

- The bound is very loose when the system is "stable". It can not differentiate between a stable and not stable systems.

Example:

- suppose we want to have a bound on solution to $\dot{Y} = -Y + \varepsilon \sin(Y)$, $Y(0) = Y_0$

- we consider the system

$$\dot{X} = -X, \qquad X(0) = Y_0$$

- Then, by applying the thm.

$$|Y(t) - X(t)| \leq \frac{\varepsilon}{L}\left(e^{tL} - 1\right)$$

where $L = 1$ is Lip. constant.

- using $X(t) = Y_0 e^{-t}$, we have

$$|Y(t) - Y_0 e^{-t}| \leq \varepsilon\left(e^{t} - 1\right)$$

$$\Rightarrow \quad Y(t) \leq Y_0 e^{-t} + \varepsilon\left(e^{t} - 1\right)$$

- The bounds obtained in thm are loose for stable sys, and blow up as $t \to \infty$

- Another useful tool is "Alekseev" formula when the nominal system is "stable".

nominal $\quad \dot{X} = f(t, X) \quad , X(0) = X_0$

perturbed $\quad \dot{Y} = f(t, Y) + g(t, Y) \quad , Y(0) = Y_0$

- Assume nominal sys. is stable, in the sense that, starting from two initial cond. $X_0, \tilde{X}_0$

$$\|X(t) - \tilde{X}(t)\| \leq c e^{-\lambda t} \|X_0 - \tilde{X}_0\| \quad \text{where } \lambda > 0$$

- Then,

$$\|Y(t) - X(t)\| \leq c e^{-\lambda t} \|Y_0 - X_0\| + \frac{c\mu}{\lambda} (1 - e^{-\lambda t})$$

Example:

$$\dot{x} = -x \qquad , \; x(0) = x_0$$

$$\dot{y} = -y + \varepsilon \sin(y) \quad , \; y(0) = y_0$$

- nominal sys. is stable, starting from $x_0, \tilde{x}_0$

$$|x(t) - \tilde{x}(t)| = |e^{-t} x_0 - e^{-t} \tilde{x}_0| \leq e^{-t} |x_0 - \tilde{x}_0|$$

- Applying the result from Alekseev formula

$$|x(t) - y(t)| \leq e^{-t} |x_0 - y_0| + \varepsilon (1 - e^{-t})$$

- Much better bound compared to

$$|x(t) - y(t)| \leq e^{+t} |x_0 - y_0| + \varepsilon (e^{t} - 1)$$

Solution to linear system using transition matrix

- Consider,

$$\dot{X}(t) = A(t)X(t) + u(t), \quad X(t_0) = X_0$$

- Define transition matrix

$$\frac{d}{dt}\phi(t,s) = A(t)\phi(t,s), \quad \phi(s,s) = I$$

$$\Longrightarrow \quad X(t) = \phi(t,t_0)X_0 + \int_{t_0}^{t}\phi(t,s)u(s)\,ds$$

- Special case : 1-dim

$$\dot{x}(t) = a(t)x(t) + u(t)$$

$$\Longrightarrow \quad \phi(t,s) = e^{\int_s^t a(\tau)d\tau}$$

$$\Longrightarrow \quad x(t) = e^{\int_{t_0}^{t} a(s)ds}x_0 + \int_{t_0}^{t} e^{\int_s^t a(\tau)d\tau}u(s)\,ds$$